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Optimum chemical balance weighing designs based on balanced bipartite block designs

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SUMMARY

The paper is studying the problem of estimating individual weights of objects, using a chemical balance weighing design under the restriction on the number of times in which each object is weighed. A lower bound for the variance of each of the estimated weights from this chemical balance weighing design is obtained and a necessary and sufficient condition for this lower bound to be attained is given. The incidence matrices of balanced bipartite block designs are used to construct the design matrix of chemical balance weighing designs under the restriction on the number in which each object is weighed.

Key words: chemical balanced weighing design, balanced bipartite block design.

1. Introduction

The results of n weighing operations to determine the individual weights of p objects with a balance that is corrected for bias will fit into the linear model

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}, \tag{1}$$

where $\mathbf{y}$ is an $n \times 1$ random column vector of the observed weights, $\mathbf{X} = (x_{ij})$, $i = 1, 2 \ldots, n$, $j = 1, 2 \ldots, p$, is an $n \times p$ matrix of known elements with $x_{ij} = -1, 1$ or 0 if the jth object is kept on the right pan, left pan, or is not included in the particular weighing operation, respectively, $\mathbf{w}$ is the $p \times 1$ column vector representing the unknown weights of objects and $\mathbf{e}$ is an $n \times 1$ random column vector of errors such that $E(\mathbf{e}) = \mathbf{0}_n$ and $E(\mathbf{e}\mathbf{e}') = \sigma^2 \mathbf{I}_n$, where $\mathbf{0}_n$ is the $n \times 1$ column vector with zero elements everywhere, $\mathbf{I}_n$ is the $n \times n$ identity matrix, "E" stands for the expectation and $\mathbf{e}'$ is used for transpose of $\mathbf{e}$.

The normal equations estimating $\mathbf{w}$ are of the form

$$\mathbf{X}'\mathbf{X}\hat{\mathbf{w}} = \mathbf{X}'\mathbf{y}, \tag{2}$$

where $\hat{\mathbf{w}}$ is the vector of the weights estimated by the least squares method.

A chemical balance weighing design is said to be singular or nonsingular, depending on whether the matrix $\mathbf{X}'\mathbf{X}$ is singular or nonsingular, respectively. It is obvious that the matrix $\mathbf{X}'\mathbf{X}$ is nonsingular if and only if the matrix $\mathbf{X}$ is of full column rank $(= p)$.

Now, if $\mathbf{X}'\mathbf{X}$ is nonsingular, the least squares estimate of $\mathbf{w}$ is given by

$$\hat{\mathbf{w}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \tag{3}$$

and the variance - covariance matrix of $\hat{\mathbf{w}}$ is

$$\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}. \tag{4}$$

Various aspects of chemical balance weighing designs have been studied by Raghavarao (1971) and Banerjee (1975). Hotelling (1944) have showed that the minimum attainable variance for each of the estimated weights for a chemical balance weighing design is σ^2/n and proved the theorem that each of the variance of the estimated weights attains the minimum if and only if $\mathbf{X}'\mathbf{X} = n\mathbf{I}_p$. This design is said to be optimum chemical balance weighing design. In other words, matrix $\mathbf{X}$ of an optimum chemical balance weighing design has as elements only -1 and 1. In this case several methods of constructing optimum chemical balance weighing design are available in the literature.

Some methods of constructing chemical balance weighing design in which the estimated weights are uncorrelated, in the case when the design matrix $\mathbf{X}$ has elements -1, 1 and 0, were given by Swamy (1982), Ceranka et al. (1998) and Ceranka and Katulska (1999).

In the present paper we study another method of constructing the design matrix $\mathbf{X}$ for an optimum chemical balance weighing design, which has elements equal to -1, 0 and 1, under the restriction on the number of times in which each object is weighed. This method is based on incidence matrices of balanced bipartite block designs.

2. Variance limit of estimated weights

Let $\mathbf{X}$ be an $n \times p$ matrix of rank p of a chemical balance weighing design and let m_j be the number of times in which jth object is weighed (i.e. the m_j be the number of elements equal to -1 and 1 in jth column of matrix $\mathbf{X}$), $j = 1, 2, ..., p$. Ceranka and Graczyk (2001) proved the following theorems.

THEOREM 2.1. *For any nonsingular chemical balance weighing design given by matrix* $\mathbf{X}$, *the variance of* $\hat{w}_j$ *of a particular* j *such that* $1 \leq j \leq p$ *cannot be less than* σ^2/m, *where* $m = max\{m_1, m_2, ..., m_p\}$.

THEOREM 2.2. *For any* $n \times p$ *matrix* $\mathbf{X}$, *of a nonsingular chemical balance weighing design, in which maximum number of elements equal to -1 and 1 in columns is equal to* m, *each of the variances of the estimated weights attains the minimum if and only if*

$$\mathbf{X}'\mathbf{X} = m\mathbf{I}_p. \tag{5}$$

DEFINITION 2.1. A nonsingular chemical balance weighing design is said to be optimal for the estimating individual weights of objects if the variances of their estimators attain the lover bound given by Theorem 2.1, i.e., if

$$Var(\hat{w}_j) = \frac{\sigma^2}{m}, j = 1, 2, ..., p. \tag{6}$$

In other words, an optimum design is given by $\mathbf{X}$ satisfying (5).

In the next sections we will present construction of matrix $\mathbf{X}$ of optimum chemical balance weighing design based on the set of the incidence matrices of balanced bipartite block designs.

3. Balanced bipartite block designs

A balanced bipartite block design is an arrangement of v treatments into b blocks such that each block containing k distinct treatments is divided into 2 subblocks containing k_1 and k_2 treatments, respectively, where $k = k_1 + k_2$. Each treatment appears in r blocks. Every pair of treatments with different subblocks appears together in λ_1 blocks and every pair of treatments with the same subblock appears together in λ_2 blocks. The integers $v, b, r, k_1, k_2, \lambda_1, \lambda_2$ are called the parameters of the balanced bipartite block design. Let $\mathbf{N}^*$ be the incidence matrix of this design.The parameters are not all independent and they are related by the following indentities

$$\begin{aligned} vr &= bk, \\ b &= \lambda_1 v(v-1)/(2k_1k_2), \\ \lambda_2 &= \lambda_1[k_1(k_1-1) + k_2(k_2-1)]/(2k_1k_2), \\ r &= \lambda_1 k(v-1)/(2k_1k_2), \\ \mathbf{N}^*\mathbf{N}^{*'} &= (r - \lambda_1 - \lambda_2)\mathbf{I}_v + (\lambda_1 + \lambda_2)\mathbf{1}_v\mathbf{1}'_v, \end{aligned} \tag{7}$$

where $\mathbf{1}_v$ is $v \times 1$ vector with elements equal to 1 everywhere.

4. Construction of design matrix using two balanced bipartite block designs

Let $\mathbf{N}_h^*$, $h = 1, 2$, be the incidence matrix of the balanced bipartite block design with the parameters $v, b_h, r_h, k_{1h}, k_{2h}, \lambda_{1h}, \lambda_{2h}$. Now, the $\mathbf{N}_h$ can be obtained from $\mathbf{N}_h^*$ by replacing the k_{1h} elements equal to 1 of each column which correspond to the elements belonging to the first subblock by –1. Thus each column of $\mathbf{N}_h$ will contain k_{1h} elements equal to –1, k_{2h} elements equal to 1 and $v - k_{1h} - k_{2h}$ elements equal to 0.

Now, we define the matrix $\mathbf{X}$ of chemical balance weighing design as

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}_1' \\ \mathbf{N}_2' \end{bmatrix}. \tag{8}$$

In this design each of the $p = v$ objects is weighed $m = r_1 + r_2$ times in $n = b_1 + b_2$ weighing operations. In other words, in each of the b_h weighing operations, $k_{1h} + k_{2h}$ objects are weighed, in such a way that k_{1h} of them are weighed on the right pan and k_{2h} on the left pan, $h = 1, 2$.

LEMMA 4.1. *The chemical balance weighing design with design matrix* $\mathbf{X}$ *given in the form (8) is nonsingular if and only if*

$$k_{11} \neq k_{21} \tag{9}$$

or

$$k_{12} \neq k_{22}. \tag{10}$$

Proof. For the design matrix $\mathbf{X}$ given above, we have

$$\mathbf{X}'\mathbf{X} = [r_1 + r_2 - (\lambda_{21} + \lambda_{22} - \lambda_{11} - \lambda_{12})]\,\mathbf{I}_v + (\lambda_{21} + \lambda_{22} - \lambda_{11} - \lambda_{12})\mathbf{1}_v\mathbf{1}_v' \tag{11}$$

and

$$\begin{aligned} \det(\mathbf{X}'\mathbf{X}) &= [r_1 + r_2 - (\lambda_{21} + \lambda_{22} - \lambda_{11} - \lambda_{12})]^{v-1} \times \\ &\quad \times [r_1 + r_2 + (v-1)(\lambda_{21} + \lambda_{22} - \lambda_{11} - \lambda_{12})]. \end{aligned} \tag{12}$$

The determinant (12) is equal to 0 if and only if

$$r_1 + r_2 = \lambda_{21} + \lambda_{22} - \lambda_{11} - \lambda_{12} \tag{13}$$

or

$$r_1 + r_2 = (1 - v)(\lambda_{21} + \lambda_{22} - \lambda_{11} - \lambda_{12}). \tag{14}$$

Using (7) it can be shown that (13) implies

$$v\left[\frac{\lambda_{11}(k_{11}+k_{21})}{2k_{11}k_{21}}+\frac{\lambda_{12}(k_{12}+k_{22})}{2k_{12}k_{22}}\right]=\frac{\lambda_{21}(k_{21}-k_{11})^2}{2k_{11}k_{21}}+\frac{\lambda_{12}(k_{22}-k_{12})^2}{2k_{12}k_{22}}$$

which is not satisfied since $v \geq k_{11}+k_{21}$ and $v \geq k_{12}+k_{22}$. Again, using (7) it can be established that (14) implies

$$(k_{21}-k_{11})^2=0$$

and

$$(k_{22}-k_{12})^2=0.$$

The last exspression is positive if and only if $k_{11} \neq k_{21}$ or $k_{12} \neq k_{22}$. □

THEOREM 4.1. *The nonsingular chemical balance weighing design with matrix* **X** *given by (8) is optimal if and only if*

$$(\lambda_{21}-\lambda_{11})+(\lambda_{22}-\lambda_{12})=0. \tag{15}$$

Proof. From the conditions (5) and (11) it follows that a chemical balance weighing design is optimal if and only if the condition (15) holds. Hence the theorem. □

If the chemical balance weighing design given by matrix **X** of the form (8) is optimal then

$$\mathrm{Var}(\hat{w}_j)=\frac{\sigma^2}{r_1+r_2}, \qquad j=1,2,\ldots,p.$$

COROLLARY 4.1. *If the nonsingular chemical balance weighing design with matrix* **X** *given by (8) is optimal then*

$$k_{21}-k_{11}=\sqrt{k_{11}+k_{21}} \tag{16}$$

and

$$k_{22}-k_{12}=\sqrt{k_{12}+k_{22}}. \tag{17}$$

Proof. According to Theorem 2.2 **X** is the matrix of the optimum chemical balance weighing design if and only if $\mathbf{X}'\mathbf{X}=m\mathbf{I}_p$. This condition is satisfied if $\lambda_{11}=\lambda_{21}$ and $\lambda_{12}=\lambda_{22}$. From (7) and (15) we have

$$\lambda_{2h}=\frac{\lambda_{1h}[k_{1h}(k_{1h}-1)+k_{2h}(k_{2h}-1)]}{2k_{1h}k_{2h}}, \quad h=1,2$$

and

$$\frac{\lambda_{11}}{2k_{11}k_{21}}[(k_{21}-k_{11})^2-(k_{11}+k_{21})]+\frac{\lambda_{12}}{2k_{12}k_{22}}[(k_{22}-k_{12})^2-(k_{12}+k_{22})]=0.$$

The thesis of corollary is a result of this equation. □

COROLLARY 4.2. *Let k_{11}, k_{21}, k_{12} and k_{22} are positive integers. The conditions (16) and (17) are satisfied if and only if*

$$k_{11}=\frac{s(s-1)}{2},\quad k_{21}=\frac{s(s+1)}{2},\quad k_{12}=\frac{u(u-1)}{2},\quad \text{and}\quad k_{22}=\frac{u(u+1)}{2},$$

where s, u are positive integers greater than 1.

From Corollary 4.2 and the relations (7) between parameters of balanced bipartite block designs we have

COROLLARY 4.3. *The existence of balanced bipartite block designs with parameters*

$$\begin{gathered} v,\ b_1=\frac{2\lambda_{11}v(v-1)}{s^2(s^2-1)},\quad r_1=\frac{2\lambda_{11}(v-1)}{s^2-1},\quad k_{11}=\frac{s(s-1)}{2}, \\ k_{21}=\frac{s(s+1)}{2},\quad \lambda_{11}=\lambda_{21} \end{gathered} \tag{18}$$

and

$$\begin{gathered} v,\ b_2=\frac{2\lambda_{12}v(v-1)}{u^2(u^2-1)},\quad r_2=\frac{2\lambda_{12}(v-1)}{u^2-1},\quad k_{12}=\frac{u(u-1)}{2}, \\ k_{22}=\frac{u(u+1)}{2},\quad \lambda_{12}=\lambda_{22} \end{gathered} \tag{19}$$

implies the existence of the optimum chemical balance weighing design with matrix $\mathbf{X}$ given in the form (8), where s, u are positive integers greater than 1.

We have seen in Corollary 4.3 that if parameters of balanced bipartite block designs are defined by (18) and (19) then a chemical balance weighing design with matrix $\mathbf{X}$ given by (8) is optimal. On the base of the series of parameters of balanced bipartite block designs given by Ceranka and Katulska (1999) we give the parameters of balanced bipartite block designs, which satisfied conditions (18) and (19) in the Table 1. These designs lead to the optimum chemical balance weighing design with matrix $\mathbf{X}$ given in the form (8). The parameters are given under the restriction $v \le 25$, $b \le 50$ and $k_{1h}+k_{2h}=4,9,16$, $h=1,2$.

We can notice, that if parameters of two balanced bipartite block designs satisfy the condition (15), then the chemical balance weighing design with matrix $\mathbf{X}$ given by (8) is optimal. We have formulated theorem following from the paper Huang (1976). Parameters of balanced bipartite block designs given in the Theorem 4.2 satisfied the condition (15).

Table 1. Parameters of balanced bipartite block designs

1 design							2 design						
v	b_1	r_1	k_{11}	k_{21}	λ_{11}	λ_{21}	v	b_2	r_2	k_{12}	k_{22}	λ_{12}	λ_{22}
5	10	8	1	3	3	3	5	10	8	1	3	3	3
5	10	8	1	3	3	3	5	20	16	1	3	6	6
5	20	16	1	3	6	6	5	20	16	1	3	6	6
6	15	10	1	3	3	3	6	15	10	1	3	3	3
6	15	10	1	3	3	3	6	30	20	1	3	6	6
6	30	20	1	3	6	6	6	30	20	1	3	6	6
7	7	4	1	3	1	1	7	7	4	1	3	1	1
7	7	4	1	3	1	1	7	14	8	1	3	2	2
7	7	4	1	3	1	1	7	21	12	1	3	3	3
7	7	4	1	3	1	1	7	28	16	1	3	4	4
7	7	4	1	3	1	1	7	35	20	1	3	5	5
7	14	8	1	3	2	2	7	14	8	1	3	2	2
7	14	8	1	3	2	2	7	21	12	1	3	3	3
7	14	8	1	3	2	2	7	28	16	1	3	4	4
7	14	8	1	3	2	2	7	35	20	1	3	5	5
7	21	12	1	3	3	3	7	21	12	1	3	3	3
7	21	12	1	3	3	3	7	28	16	1	3	4	4
7	21	12	1	3	3	3	7	35	20	1	3	5	5
7	28	16	1	3	4	4	7	28	16	1	3	4	4
7	28	16	1	3	4	4	7	35	20	1	3	5	5
7	35	20	1	3	5	5	7	35	20	1	3	5	5
8	28	14	1	3	3	3	8	28	14	1	3	3	3
9	36	16	1	3	3	3	9	36	16	1	3	3	3
10	10	9	3	6	4	4	10	10	9	3	6	4	4
10	10	9	3	6	4	4	10	15	6	1	3	1	1
10	10	9	3	6	4	4	10	20	18	3	6	8	8
10	10	9	3	6	4	4	10	30	12	1	3	2	2
10	10	9	3	6	4	4	10	45	18	1	3	3	3
10	15	6	1	3	1	1	10	15	6	1	3	1	1
10	10	9	3	6	4	4	10	30	12	1	3	2	2

THEOREM 4.2. *The existence of a balanced bipartite block designs with the parameters*
(i) $v = 6s, b_1 = 6s(6s-1), r_1 = 3(6s-1), k_{11} = 1, k_{21} = 2, \lambda_{11} = 4, \lambda_{21} = 2$ *and* $v = 6s$, $b_2 = 6s(6s-1)$, $r_2 = 7(6s-1)$, $k_{12} = 2$, $k_{22} = 5$, $\lambda_{12} = 20$, $\lambda_{22} = 22$, $s = 2, 3, \ldots,$
(ii) $v = 4s+1, b_1 = 2s(4s+1), r_1 = 16s, k_{11} = 3, k_{21} = 5, \lambda_{11} = 15, \lambda_{21} = 13$ *and* $v = 4s+1, b_2 = s(4s+1), r_2 = 8s, k_{12} = 2, k_{22} = 6, \lambda_{12} = 6, \lambda_{22} = 8,\ s = 2, 3, \ldots,$
(iii) $v = 4s+1,\ b_1 = 2s(4s+1),\ r_1 = 6s,\ k_{11} = 1, k_{21} = 2,\ \lambda_{11} = 2,\ \lambda_{21} = 1$ *and* $v = 4s+1, b_2 = s(4s+1), r_2 = 5s, k_{12} = 1, k_{22} = 4, \lambda_{12} = 2, \lambda_{22} = 3,\ s = 1, 2, \ldots,$

Table 1. Continued

1 design							2 design						
v	b_1	r_1	k_{11}	k_{21}	λ_{11}	λ_{21}	v	b_2	r_2	k_{12}	k_{22}	λ_{12}	λ_{22}
10	10	9	3	6	4	4	10	45	18	1	3	3	3
10	15	6	1	3	1	1	10	15	6	1	3	1	1
10	15	6	1	3	1	1	10	20	18	3	6	8	8
10	15	6	1	3	1	1	10	30	12	1	3	2	2
10	15	6	1	3	1	1	10	45	18	1	3	3	3
10	20	18	3	6	8	8	10	20	18	3	6	8	8
10	20	18	3	6	8	8	10	30	12	1	3	2	2
10	20	18	3	6	8	8	10	45	18	1	3	3	3
10	30	12	1	3	2	2	10	30	12	1	3	2	2
10	30	12	1	3	2	2	10	45	18	1	3	3	3
10	45	18	1	3	3	3	10	45	18	1	3	3	3
13	13	9	3	6	3	3	13	13	9	3	6	3	3
13	13	9	3	6	3	3	13	26	8	1	3	1	1
13	13	9	3	6	3	3	13	26	18	3	6	6	6
13	26	8	1	3	1	1	13	26	8	1	3	1	1
13	26	8	1	3	1	1	13	26	18	3	6	6	6
13	26	18	3	6	6	6	13	26	18	3	6	6	6
16	40	10	1	3	1	1	16	40	10	1	3	1	1
18	34	17	3	6	4	4	18	34	17	3	6	4	4
19	19	9	3	6	2	2	19	19	9	3	6	2	2
19	19	9	3	6	2	2	19	38	18	3	6	4	4
19	38	18	3	6	4	4	19	38	18	3	6	4	4
21	21	16	6	10	6	6	21	21	16	6	10	6	6
21	21	16	6	10	6	6	21	35	15	3	6	3	3
21	35	15	3	6	3	3	21	35	15	3	6	3	3
25	25	16	6	10	5	5	25	25	16	6	10	5	5
25	25	16	6	10	5	5	25	50	18	3	6	3	3
25	50	18	3	6	3	3	25	50	18	3	6	3	3

(iv) $v = 10s+1, b_1 = 5s(10s+1), r_1 = 15s, k_{11} = 1, k_{21} = 2, \lambda_{11} = 2, \lambda_{21} = 1$ and $v = 10s+1,\ b_2 = s(10s+1),\ r_2 = 6s, k_{12} = 1, k_{22} = 5,\ \lambda_{12} = 1,\ \lambda_{22} = 2, s = 2, 3, ...,$

(v) $v = 2s+1, b_1 = s(2s+1), r_1 = 3s, k_{11} = 1, k_{21} = 2, \lambda_{11} = 2, \lambda_{21} = 1$ *and* $v = 2s+1,\ b_2 = s(2s+1), r_2 = 7s, k_{12} = 2, k_{22} = 5, \lambda_{12} = 10, \lambda_{22} = 11, s = 6, 7, ...,$

(vi) $v = 4s+1, b_1 = s(4s+1), r_1 = 4s, k_{11} = 2, k_{21} = 2, \lambda_{11} = 2, \lambda_{21} = 1$ *and* $v = 4s+1, b_2 = s(4s+1), r_2 = 5s, k_{12} = 1, k_{22} = 4, \lambda_{12} = 2, \lambda_{22} = 3, s = 1, 2, ...,$

(vii) $v = 20s+1, b_1 = 5s(20s+1), r_1 = 20, k_{11} = 2, k_{21} = 2, \lambda_{11} = 2, \lambda_{21} = 1$ *and* $v = 20s+1, b_2 = 2s(20s+1), r_2 = 12s, k_{12} = 1, k_{22} = 5, \lambda_{12} = 1, \lambda_{22} = 2, s = 1, 2, ..,$

(viii) $v = 4s+1, b_1 = s(4s+1), r_1 = 4s, k_{11} = 2, k_{21} = 2, \lambda_{11} = 2, \lambda_{21} = 1$ *and* $v = 4s+1,\ b_2 = 2s(4s+1),\ r_2 = 14s,\ k_{12} = 2,\ k_{22} = 5,\ \lambda_{12} = 10,\ \lambda_{22} = 11,$ $s = 2, 3, \ldots,$
(ix) $v = 4s+1, b_1 = s(4s+1), r_1 = 5s, k_{11} = 2, k_{21} = 3, \lambda_{11} = 3, \lambda_{21} = 2$ *and* $v = 4s+1,\ b_2 = s(4s+1), r_2 = 5s, k_{12} = 1, k_{22} = 4, \lambda_{12} = 2, \lambda_{22} = 3, s = 2, 3, \ldots,$
(x) $v = 20s+1, b_1 = 5s(20s+1), r_1 = 20s, k_{11} = 2, k_{21} = 3, \lambda_{11} = 3, \lambda_{21} = 2$ *and* $v = 20s+1, b_2 = 2s(20s+1), r_2 = 12s, k_{12} = 1, k_{22} = 5, \lambda_{12} = 1, \lambda_{22} = 2, s = 1, 2, \ldots,$
(xi) $v = 4s+1, b_1 = s(4s+1), r_1 = 5s, k_{11} = 2, k_{21} = 3, \lambda_{11} = 3, \lambda_{21} = 2$ *and* $v = 4s+1, b_2 = 2s(4s+1), r_2 = 14s, k_{12} = 2, k_{22} = 5, \lambda_{12} = 10, \lambda_{22} = 11, s = 2, 3, \ldots,$
(xii) $v = 4s+1, b_1 = 2s(4s+1), r_1 = 12s, k_{11} = 2, k_{21} = 4, \lambda_{11} = 8, \lambda_{21} = 7$ *and* $v = 4s+1, b_2 = s(4s+1), r_2 = 5s, k_{12} = 1, k_{22} = 4, \lambda_{12} = 2, \lambda_{22} = 3, s = 2, 3, \ldots,$
(xiii) $v = 10s+1, b_1 = 5s(10s+1), r_1 = 30s, k_{11} = 2, k_{21} = 4, \lambda_{11} = 8, \lambda_{21} = 7$ *and* $v = 10s+1,\ b_2 = s(10s+1),\ r_2 = 6s,\ k_{12} = 1, k_{22} = 5, \lambda_{12} = 1,\ \lambda_{22} = 2, s = 1, 2, \ldots,$
(xiv) $v = 2s+1, b_1 = s(2s+1), r_1 = 6s, k_{11} = 2, k_{21} = 4, \lambda_{11} = 8, \lambda_{21} = 7$ *and* $v = 2s+1, b_2 = s(2s+1), r_2 = 7s, k_{12} = 2, k_{22} = 5, \lambda_{12} = 10, \lambda_{22} = 11, s = 3, 4, \ldots,$
implies the existence of the optimum chemical balance weighing design with matrix $\mathbf{X}$ *given in the form (8).*

Proof. It is easy to prove that parameters of balanced bipartite block designs satisfy the condition (15). □

5. Optimum chemical balance weighing design based on any number of balanced bipartite block designs

Let $\mathbf{N}_h^*$, $h = 1, 2, \ldots t$, be the incidence matrix of the balanced bipartite block designs with the parameters $v, b_h, r_h, k_{1h}, k_{2h}, \lambda_{1h}, \lambda_{2h}$. From $\mathbf{N}_h^*$ we obtain other matrices $\mathbf{N}_h$ by replacing the k_{1h} elements equal to 1 of each column which correspond to the elements belonging to the first subblock by −1. Thus each column of $\mathbf{N}_h$ will contain k_{1h} elements equal to −1, k_{2h} elements equal to 1 and $v - k_{1h} - k_{2h}$ elements equal to 0. In other words, in each of the b_h weighing operations $k_{1h} + k_{2h}$ objects are weighed. Among them k_{1h} are weighed on the right and k_{2h} on the left pan.

Now, we define the matrix $\mathbf{X}$ of the chemical balance weighing design as

$$\mathbf{X}' = [\mathbf{N}_1 \vdots \mathbf{N}_2 \vdots \ldots \vdots \mathbf{N}_t]. \tag{20}$$

In this design each of the $p = v$ objects is weighed $m = \sum_{h=1}^{t} r_h$ times in $n = \sum_{h=1}^{t} b_h$ weighing operations.

LEMMA 5.1. *The chemical balance weighing design with design matrix* $\mathbf{X}$ *given in the form (20) is nonsingular if and only if*

$$k_{1h} \neq k_{2h} \tag{21}$$

for at least one h, $h = 1, 2, ...t$.

Proof. For the design matrix $\mathbf{X}$ given above, we have

$$\mathbf{X}'\mathbf{X} = \left[\sum_{h=1}^{t}(r_h - \lambda_{2h} + \lambda_{1h})\right]\mathbf{I}_v + \left[\sum_{h=1}^{t}(\lambda_{2h} - \lambda_{1h})\right]\mathbf{1}_v\mathbf{1}_v' \tag{22}$$

and

$$\det(\mathbf{X}'\mathbf{X}) = \left[\sum_{h=1}^{t}(r_h - \lambda_{2h} + \lambda_{1h})\right]^{v-1}\sum_{h=1}^{t}[r_h + (v-1)(\lambda_{2h} - \lambda_{1h})]. \tag{23}$$

The determinant (23) is equal to 0 if and only if

$$\sum_{h=1}^{t} r_h = \sum_{h=1}^{t}(\lambda_{2h} - \lambda_{1h}) \tag{24}$$

or

$$(1-v)\sum_{h=1}^{t}(\lambda_{2h} - \lambda_{1h}) = \sum_{h=1}^{t} r_h. \tag{25}$$

Using (7) it can be shown that (24) implies

$$\frac{v}{2}\sum_{h=1}^{t}\frac{\lambda_{1h}(k_{1h} + k_{2h})}{k_{1h}k_{2h}} = \frac{1}{2}\sum_{h=1}^{t}\frac{\lambda_{1h}(k_{2h} - k_{1h})^2}{k_{1h}k_{2h}},$$

which is not satisfied since $v \geq k_{1h} + k_{2h}$, $h = 1, 2, ..., t$. Again, using (7) it can be estabilished that (25) implies

$$(k_{1h} - k_{2h})^2 = 0 \quad \text{for each} \quad h = 1, 2, ..., t.$$

The last expression is positive if and only if $k_{1h} \neq k_{2h}$ for at least one h, $h = 1, 2, ..., t$. So lemma is proved. □

THEOREM 5.1. *The nonsingular chemical balance weighing design with matrix* $\mathbf{X}$ *given by (20) is optimal if and only if*

$$\sum_{h=1}^{t}(\lambda_{2h} - \lambda_{1h}) = 0. \tag{26}$$

Proof. From the conditions (5) and (22) it follows that a chemical balance weighing design is optimal if and only if the condition (26) holds. Hence the theorem. □

COROLLARY 5.1. *If the nonsingular chemical balance weighing design with matrix* $\mathbf{X}$ *given by (20) is optimal then*

$$k_{2h} - k_{1h} = \sqrt{k_{1h} + k_{2h}} \quad \text{for each } h = 1, 2, ..., t. \tag{27}$$

Proof. According to Theorem 2.2 $\mathbf{X}$ is the matrix of the optimum chemical balance weighing design if and only if $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p$. This condition is satisfied if $\lambda_{1h} = \lambda_{2h}$, $h = 1, 2, ..., t$. From (7) and (26) we have

$$\lambda_{2h} = \frac{\lambda_{1h}[k_{1h}(k_{1h} - 1) + k_{2h}(k_{2h} - 1)]}{2k_{1h}k_{2h}}, \quad h = 1, 2, ..., t$$

and

$$\frac{1}{2}\sum_{h=1}^{t} \frac{\lambda_{1h}[(k_{2h} - k_{1h})^2 - (k_{1h} + k_{2h})]}{k_{1h}k_{2h}} = 0.$$

The thesis of corollary is a result of this equation. □

COROLLARY 5.2. *Let* k_{1h} *and* $k_{2h}, h = 1, 2, ...t$*, are positive integers. The conditions (27) are satisfied if and only if*

$$k_{1h} = \frac{s_h(s_h - 1)}{2}$$

and

$$k_{2h} = \frac{s_h(s_h + 1)}{2},$$

where s_h*,* $h = 1, 2, ..., t$*, are positive integers greater than 1.*

From Theorem 5.1 we have

COROLLARY 5.3. *The existence of balanced bipartite block designs with parameters* $v, b_h, r_h, k_{1h}, k_{2h}, \lambda_{1h}, \lambda_{2h}, h = 1, 2, ..., t$*, for which condition (26) holds implies the existence of the optimum chemical balance weighing design with matrix* $\mathbf{X}$ *given in the form (20).*

There is a big number of combinations between parameters of balanced bipartite block designs for which condition (26) holds. Thus, we have many possible constructions of the matrix $\mathbf{X}$ of optimal chemical balance weighing design for given $p = v$ and $n = \sum_{h=1}^{t} b_h$.

6. Example

Let us assume that we want to estimate unknown measurements of $p = v = 5$ objects and we have at our disposal $n = 20$ measurement operations. We have to choose design matrix $\mathbf{X}$ in the form (20) in such a manner that the variance of each estimator of the unknown measurement of object attains the lower bound, i.e. $\mathrm{Var}(\hat{w}_j) = \sigma^2 / \sum_{h=1}^{t} r_h$ for each $j = 1, 2, ...p$. We assume also that each object can be measured 17 times. Then the design matrix $\mathbf{X}$ in the form (20) could be constructed from three incidence matrices of balanced bipartite block designs with parameters $v = 5$, $b_1 = 10$, $r_1 = 8$, $k_{11} = 3$, $k_{21} = 1$, $\lambda_{11} = 3$, $\lambda_{21} = 3$, the second one $v = 5$, $b_2 = 5$, $r_2 = 4$, $k_{12} = 2$, $k_{22} = 2$, $\lambda_{12} = 2$, $\lambda_{22} = 1$ and $v = 5$, $b_3 = 5$, $r_3 = 5$, $k_{13} = 1$, $k_{23} = 4$, $\lambda_{13} = 2$, $\lambda_{23} = 3$. In this design $\mathrm{Var}(\hat{w}_j) = \frac{\sigma^2}{17}$, $j = 1, 2, ...p$.

The design matrix $\mathbf{X}$ given in the form (20) is optimal for each $t = 1, 2, 3$, but from these three possible constructions we can choose the best matrix $\mathbf{X}$. According to the definition 2.1 the best one would be construction for $t = 3$. Based on the paper of Huang (1976) we construct the incidence matrices of balanced bipartite block designs $v = 5$, $b_1 = 10$, $r_1 = 8$, $k_{11} = 1$, $k_{21} = 3$, $\lambda_{11} = 3$, $\lambda_{21} = 3$

$$\mathbf{N}_1 = \begin{bmatrix} 0 & 1_2 & 1_2 & 1_2 & 1_1 & 0 & 1_2 & 1_1 & 1_2 & 1_2 \\ 1_1 & 0 & 1_2 & 1_2 & 1_2 & 1_2 & 0 & 1_2 & 1_1 & 1_2 \\ 1_2 & 1_1 & 0 & 1_2 & 1_2 & 1_2 & 1_2 & 0 & 1_2 & 1_1 \\ 1_2 & 1_2 & 1_1 & 0 & 1_2 & 1_1 & 1_2 & 1_2 & 0 & 1_2 \\ 1_2 & 1_2 & 1_2 & 1_1 & 0 & 1_2 & 1_1 & 1_2 & 1_2 & 0 \end{bmatrix},$$

where 1_1 and 1_2 denote that the object exists in the first or in second subblock, respectively, 0 the object does not exist in the block, $v = 5$, $b_2 = 5$, $r_2 = 4$, $k_{12} = 2$, $k_{22} = 2$, $\lambda_{12} = 2$, $\lambda_{22} = 1$,

$$\mathbf{N}_2 = \begin{bmatrix} 1_1 & 1_1 & 1_2 & 1_1 & 0 \\ 1_1 & 1_2 & 1_1 & 0 & 1_2 \\ 1_2 & 1_1 & 0 & 1_2 & 1_2 \\ 1_2 & 0 & 1_1 & 1_1 & 1_1 \\ 0 & 1_2 & 1_2 & 1_2 & 1_1 \end{bmatrix},$$

$v = 5$, $b_3 = 5$, $r_3 = 5$, $k_{13} = 1$, $k_{23} = 4$, $\lambda_{13} = 2$, $\lambda_{23} = 3$

$$\mathbf{N}_3 = \begin{bmatrix} 1_1 & 1_2 & 1_2 & 1_2 & 1_2 \\ 1_2 & 1_1 & 1_2 & 1_2 & 1_2 \\ 1_2 & 1_2 & 1_1 & 1_2 & 1_2 \\ 1_2 & 1_2 & 1_2 & 1_1 & 1_2 \\ 1_2 & 1_2 & 1_2 & 1_2 & 1_1 \end{bmatrix}.$$

In each incidence matrix of balanced bipartite block design we replace the elements, that are equal to 1 and they correspond to the elements belonging to the first subblock, (1_1) by -1. As the next step we built design matrix $\mathbf{X}$ in the form (20) for $t = 3$

$$\mathbf{X} = \begin{bmatrix} 0 & -1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 & -1 \\ -1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 1 & 0 \\ -1 & -1 & 1 & 1 & 0 \\ -1 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ -1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix}.$$

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Optymalne chemiczne układy wagowe oparte na dwudzielnych układach bloków

STRESZCZENIE

W pracy przedstawiony jest problem estymacji indywidualnych miar obiektów przy wykorzystaniu modelu chemicznego układu wagowego, dodatkowo zakładając ograniczenia na liczbę pomiarów. Podane zostało dolne ograniczenie na wariancję estymatora każdej miary obiektów oraz warunki konieczne i dostateczne na to, aby wariancja estymatora osiągała to dolne ograniczenie. Do konstrukcji macierzy układu optymalnego chemicznego układu wagowego wykorzystujemy macierze incydencji dwudzielnych układów bloków.

Słowa kluczowe: chemiczny układ wagowy, dwudzielny układ bloków.